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WHAT LIES BETWEEN + AND × (and beyond)?

H.P.Williams

London School of Economics

h.p.williams@lse.ac.uk

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London School of Economics and Political Science

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Abstract

We attempt to create a continuum of functions between addition and multiplication (and beyond). Such a function could have practical applications. Addition, multiplication, exponentiation, tetration etc. are all particular cases of a generalisation of Ackermann's function for successive integral values of one of the arguments. Intermediate functions can be viewed as results of a fractional value of this argument. Ways of seeking such functions with given properties are investigated. It is noted that some other integrally defined functions of mathematics can be extended to fractional arguments. Two possible approaches, which are considered here, are (i) to consider Gauss's Arithmetic-Geometric mean and (ii) to consider solutions of the functional equation $ff(x) = e^x$.

Keywords: Ackermann's function, tetration, Gauss's mean, functional equations.

1. INTRODUCTION

Multiplication can be regarded as successive addition, exponentiation as successive multiplication etc. The next operation (tetration) requires a stipulation about the order in which successive exponentiations are carried out (since exponentiation is not commutative). We therefore consider the hierarchy of functions

$$a+b$$

$$a \times b = a + a + \ldots a \quad \text{(b times)}$$

$$a^b = a \times a \times \ldots \times a \quad \text{(b times)}$$

$${}^{b}a = a^{a^{\cdot^{\cdot^{a}}}} \quad \text{(b times)}$$

etc.

The fourth operation is usually known as 'tetration' and sometimes written as indicated. The convention is adopted of assuming bracketing from the top.The name was coined by Goodstein [4]. It has received some attention (see the web site of Geisler [3]). In particular defining the function for fractional b presents a problem, as discussed below.

The heirarchy of these functions has also been considered by Rubtsov and Romerio [5].

We could introduce the successor function a + 1 as the 'zeroth' function if we wished.

In order to motivate the problem we will consider a numerical example.

$$2 + 3 = 5$$

$$2 \times 3 = 6$$

$$2^3 = 8$$

$${}^{3}2 = 16$$

Marked on the graph in figure 1 it can be seen that it is possible to draw a smooth curve though these values where r = 1, 2, 3, 4 for addition, multiplication, exponentiation, tetration respectively. In order to seek the value for an operation halfway between + and × we seek the value at the point indicated by the arrow.

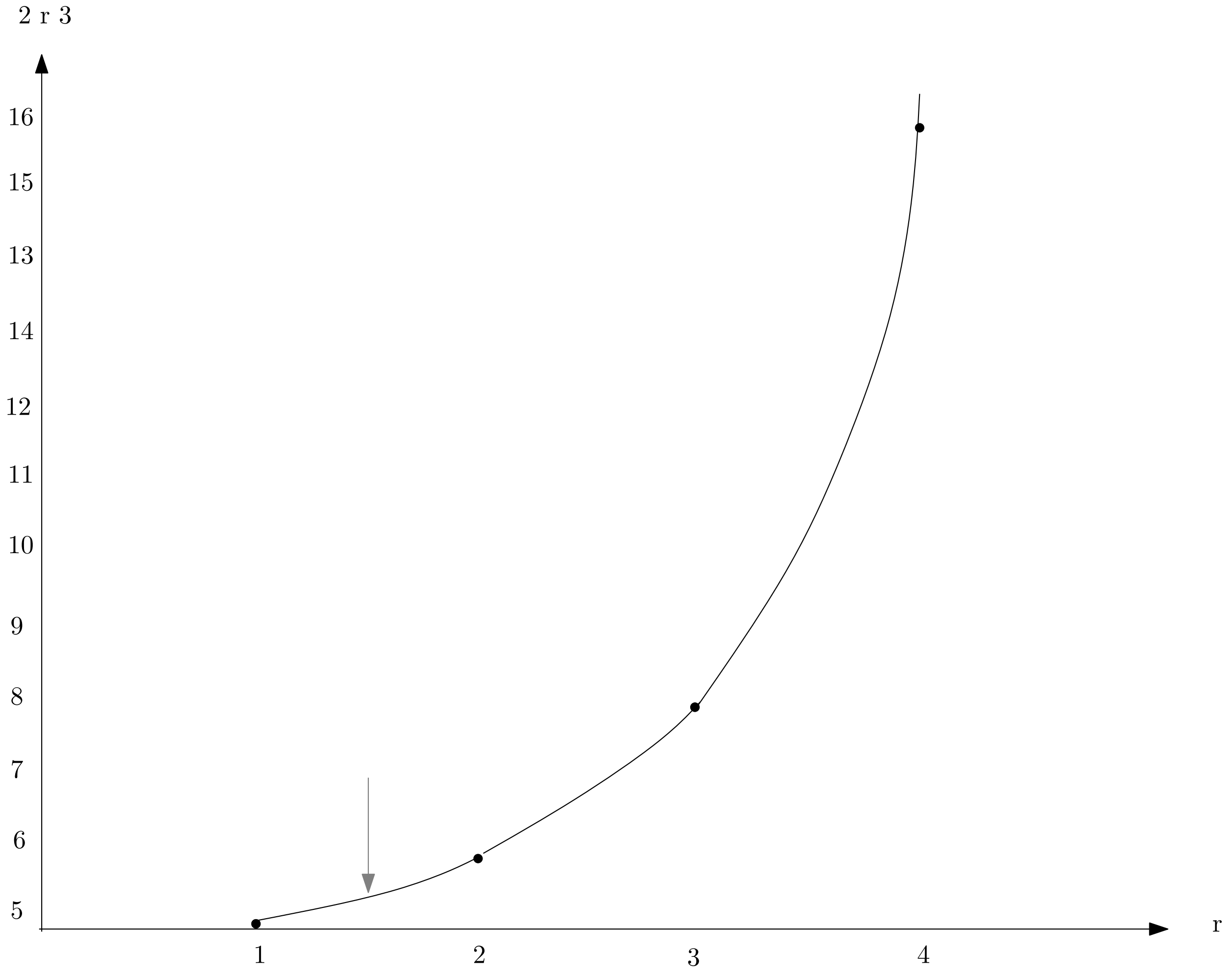


Figure 1. Successive Arithmetic Operations Applied to 2 and 3

2. ACKERMANN'S FUNCTION

A generalisation of Ackermann's function [1] can be defined easily by the following recursion.

$$f(a+1, b+1, c) = f(a, f(a+1,b,c), c)$$

with initial conditions

$$f(0, b, c) = b+1$$

$$f(1, 0, c) = c$$

$$f(2, 0,c) = 0$$

$$f(a+1, 0, c) = 1 \text{ for } a> 1$$

It is easy to verify that

$f(0, b, c) = b+1$ the successor function

$f(1, b, c) = b+c$ addition

$f(2, b, c) = b x c$ multiplication

$f(3, b, c) = c^b$ exponentiation

$f(4, b, c) = {}^{b}c$ tetration

We seek $f({}^{3}/_{2}, b, c)$.

Ackermann's function is usually expressed as a function of 2 arguments by fixing c at (say) 2. It is a doubly recursive function which grows faster than any primitive recursive function: eg. in order to evaluate f(a+1, . , .) we need to evaluate f(a+1, .,.) for smaller arguments and f(a, . , .) for much larger arguments. Its 'explosive' growth is demonstrated by

$$f(0, 3, 2) = 4$$

$$f(1, 3, 2) = 5$$

$$f(2, 3, 2) = 6$$

$$f(3, 3, 2) = 8$$

$$f(4, 3, 2) = 16$$

$$f(5, 3, 2) = 65536$$

It is interesting to note that f(a, b, c) is not well defined for fractional b either. Eg. what is

(½)2 ?

We can, however, define $({}^{1}/\infty)\ 2$. It is √2 since ${}^{\infty}\sqrt{2} = 2$.

3. GAUSS'S ARITHMETIC-GEOMETRIC MEAN

Let A(a, b) = (a+b)/2 the arithmetic mean

G(a,b) = √(a x b) the geometric mean

M(a, b) , Gauss's Arithmetic-Geometric mean (see eg. Cox [2]) is 'halfway between' A(a, b) and G(a, b) and is defined, iteratively, by

$a_1 = G(a, b)$, $b_1 = A(a, b)$

$a_{n+1} = G(a_n , b_n)$, $b_{n+1} = A(a_n , b_n)$

$M(a, b) = Lt_{n \to \infty}\ a_n = Lt_{n \to \infty}\ b_n$

For example g(2,128) = 16, A(2,128) = 65, M(2,128) = 36.26

Since a + b = A(a, b) x 2 = A(a, b) $_2$ 2

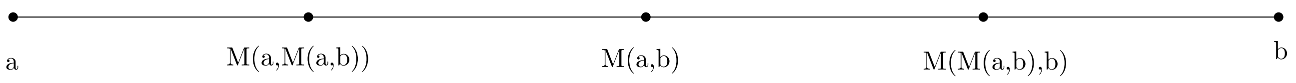
a x b = $G(a, b)^2$ = G(a, b) $_3$ 2

Let $a\ {}^{3}/_{2}\ b = M(a, b)\ {}^{5}/_{2}\ 2 = M(a, b)\ {}^{3}/_{2}\ M(a, b))$

M(a, b) has an analytic solution in terms of elliptic integrals.

But there is a difficulty.

Consider the following values on a line.



While M(a,M(a,b)) is the mean of a and M(a,b) and M(M(a,b),b) is the mean of M(a,b) and b,

M(a,b) ≠ M(M(a,M(a,b)), M(M(a,b),b)).

4. THE FUNCTIONAL EQUATION ff(x) = e^x

A ‘bridge’ between addition and multiplication is provided by the exponential function (or its inverse , the logarithmic function) since

$e^{(a+b)} = e^a \times e^b$.

Let ff(x) = e^x .

We can define f(a $^3/_2$ b) as f(a) x f(b).

Therefore we seek solutions of ff(x) = e^x. f(x) is a function ‘between’ x (ie the identity function) and e^x.

This functional equation has been examined by a number of authors. For example Hammersley [5] and subsequent correspondence in the IMA Bulletin. However we seek a solution with a number of ‘reasonable’ conditions. In particular we initially demand that f(x) (defined on the non-negative reals) satisfies

(i) $x < f(x) < e^x$.
(ii) f(x) is monotonic strictly increasing.
(iii) f(x) is continuous and infinitely differentiable.
(iv) The derivatives are monotonic strictly increasing.

Let us define f(0) =p.

Then f(p) = e^0 = 1

f(1) = e^p

f(e^p) = f(1)

This gives the following table of values

x	0	p	1	e^p	e	e^{e^p}	e^e	$e^{e^{e^p}}$	. . .
f(x)	p	1	e^p	e	e^{e^p}	e^e	$e^{e^{e^p}}$	e^{e^e}	. . .

As necessary (discrete) conditions for the above (continuous) conditions we demand that f(x), its gradients, gradients of gradients etc. are monotonic increasing and take intermediate values between the corresponding values of x and f(x). This implies

$1 < (1 - p)/p < (e^p - 1)/(1 - p)$ giving $0.469.. < p < 0.5$

It would appear that f(x) is not unique.

We plot possible values of f(x) if we set p at (say) 0.49 giving

x	0	0.49	1	1.63	2.72	5.10	15.18
f(x)	0.49	1	1.63	2.72	5.10	15.18	164.02

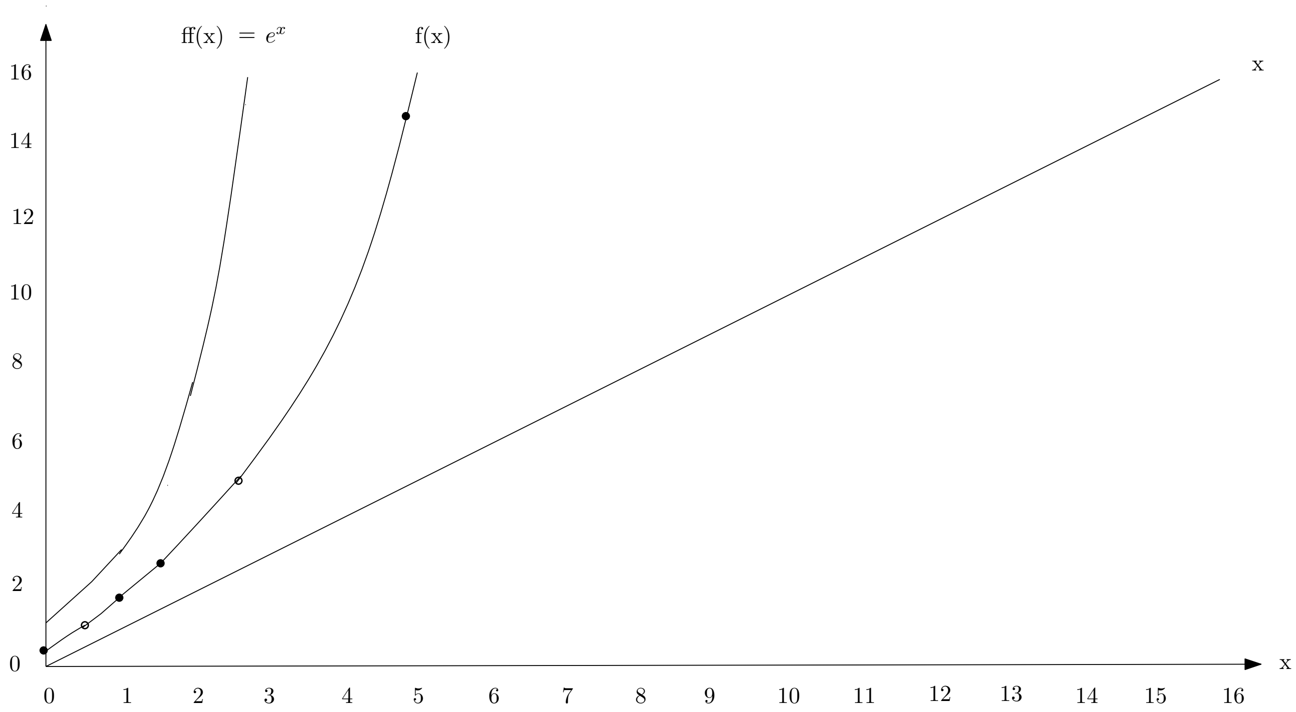


Figure 2 The functions x, f(x) and ff(x) = e^x .

5. REFERENCES

1. Ackermann, W., 1928 Zum Hilbertschen Aufbau der reellen Zahlen, *Mathematische Annalen* **99** 118-133.
2. Cox, D.A., 1985 Gauss and the geometric-arithmetic mean, *Notices Amer. Math. Soc.* **32(2)** 147-151.
3. Geisler, D., 2009 Tetration web site, http://www.tetration.org
4. Goodstein, R.L., 1947 Tranfinite ordinals in recursive number theory, *Journal of Symbolic Logic* **12.**
5. Hammersley, J.M., 1983 Functional roots and indicial semigroups, *Bulletin of the IMA* **19** 194-196.
6. Rubtsov, C.A., and Romerio, G.F., 2004 Ackermann's function and new arithmetical operations, http://www.rotarysaluzzo.it/filePD=/Iperoperazi%20(1).pdf